Braid group statistics in two-dimensional quantum field theory

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Abstract

Within the framework of algebraic quantum field theory, we construct explicitly localized morphisms of a Haag-Kastler net in $1+1$-dimensional Minkowski space showing abelian braid group statistics. Moreover, we investigate the scattering theory of the corresponding quantum fields.

Introduction

The notion of statistics has been of great importance since the early days of quantum physics, e.g. in connection with Pauli's exclusion principle or Bose condensation. Originally, one believed permutation group statistics to be the only possibility to appear; nowadays, one knows that this is true only in $d \geq 4$ space-time dimensions, while in low dimensional theories braid group statistics can occur [3], [15].

Here we consider charges localized in some double cone $\mathcal{O}$ in $1+1$-dimensional space-time, and the appearance of braid group statistics is due to the fact that the causal complement of $\mathcal{O}$ is disconnected, see e.g. [17].

One of the basic achievements of the theory of superselection sectors is to give an intrinsic definition of statistics. We use the results of the abstract analysis to investigate a concrete model showing abelian braid group statistics.

In section 1 we outline the algebraic approach to quantum field theory as initiated by Haag and Kastler; in particular, we define the statistics operator

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— which will play a crucial role in the subsequent treatment — and show how this object gives rise to a representation of the braid group.

After that, in section 2, we introduce a class of automorphisms of the algebra of canonical anticommutation relations, namely the group of Bogoliubov transformations implementable in some Fock representation, which is an essential tool in the construction of localized morphisms; in addition, we specify a provisional net of observables, which does not yet meet all necessary conditions for the general theory of superselection sectors to be applicable.

We define localized automorphisms of this net in section 3, using Bogoliubov operators studied before by Ruijsenaars in connection with certain integrable field theories [29]. Moreover, we compute the statistics and obtain a one-dimensional representation of the braid group.

In section 4 we extend our provisional net to get one fulfilling the requirements of superselection theory; furthermore, we extend the morphisms constructed in the previous section to the latter and see how this gives rise to an additional quantum number purely topological in nature.

Finally, in section 5, we discuss the Haag-Ruelle scattering theory for the corresponding quantum fields, obtaining a close relation between scalar products of scattering states and the statistics operator.

1 The algebraic approach

Besides the Wightman framework [35], algebraic quantum field theory, which has its origin in the work of Haag, Kastler and Araki [1], [18], has proven successful for the investigation of conceptional questions in quantum physics. Since we take the algebraic point of view in this article, we sketch the main features of this approach; details can be found in the book of Haag [17] or in [19] as well as in the original papers [6], [7], [8].

Algebraic quantum field theory is based on the assumption that a quantum field theory is fixed once the net of observables is specified; more precisely, denote by $\mathcal{K}$ the set of open double cones [1] in Minkowski space; to each double cone $\mathcal{O} \in \mathcal{K}$ one associates a von Neumann algebra $\mathfrak{A}(\mathcal{O})$ in a Hilbert space $\mathcal{H}$, whose self-adjoint elements are interpreted as the observables measurable within $\mathcal{O}$, such that the following properties are fulfilled (Haag-Kastler axioms):

1. Isotony:

$$\mathcal{O}_1 \subset \mathcal{O}_2 \Longrightarrow \mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2) \ , \tag{1}$$

[1]Recall that a double cone is a non-void intersection of a forward and a backward light cone.

i.e. by enlarging the space-time region the algebra of observables cannot become smaller. Since $\mathcal{K}$ is directed, one can consider the algebra

$$\mathfrak{A}_{\text{loc}} := \bigcup_{\mathcal{O}\in\mathcal{K}} \mathfrak{A}(\mathcal{O}) \tag{2}$$

of local observables, which possesses a unique C^*-norm $\|.\|_{C^*}$; by completion with respect to this norm one obtains the so-called *quasilocal algebra*

$$\mathfrak{A} := \overline{\mathfrak{A}_{\text{loc}}}^{\|.\|_{C^*}} . \tag{3}$$

Put differently: $\mathfrak{A}$ is the C^*-inductive limit of the algebras $\mathfrak{A}(\mathcal{O})$.

2. **Locality**: The observation that events in space-like separated regions cannot influence each other is taken into account by demanding commutativity for algebras associated with space-like separated double cones ($\times$ denotes space-like separation):

$$\mathcal{O}_1 \times \mathcal{O}_2 \Longrightarrow [\mathfrak{A}(\mathcal{O}_1), \mathfrak{A}(\mathcal{O}_2)] = 0 \tag{4}$$

or equivalently

$$\mathcal{O}_1 \subset \mathcal{O}_2' \Longrightarrow \mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2)' . \tag{5}$$

Here $\mathfrak{A}(\mathcal{O})'$ means the commutant of $\mathfrak{A}(\mathcal{O})$ and $\mathcal{O}'$ denotes the causal complement of $\mathcal{O}$. Since this axiom mirrors the fact that no signal can propagate faster than light it is sometimes called *Einstein causality*.

3. **Haag duality**: Locality can be strengthened by claiming $\mathfrak{A}(\mathcal{O})$ to be the maximal algebra fulfilling (5), i.e.

$$\mathfrak{A}(\mathcal{O}) = \mathfrak{A}(\mathcal{O}')' , \tag{6}$$

where $\mathfrak{A}(\mathcal{O}')$ denotes the C^*-subalgebra of $\mathfrak{A}$ generated by all $\mathfrak{A}(\mathcal{O}_1)$ with $\mathcal{O}_1 \times \mathcal{O}$. This condition is not always met; in particular, it is violated for the net of observables of the free Dirac field in two space-time dimensions, see e.g. [17].

4. **Covariance**: There exists a representation α of the Poincaré group $\mathcal{P}_+^\uparrow$ by automorphisms of $\mathfrak{A}$ such that

$$\alpha_{x,\Lambda}(\mathfrak{A}(\mathcal{O})) = \mathfrak{A}((x,\Lambda)\mathcal{O}) . \tag{7}$$

5. **Spectrum condition**: There is a strongly continuous unitary representation $(x,\Lambda) \mapsto U(x,\Lambda)$ of $\mathcal{P}_+^\uparrow$ on the physical Hilbert space $\mathcal{H}$ implementing $\alpha_{(x,\Lambda)}$, i.e.

$$\alpha_{(x,\Lambda)}(A) = U(x,\Lambda) A U(x,\Lambda)^* , \tag{8}$$

such that the generators of U have spectrum in $\overline{V_+}$ (the closure of the forward light cone) [2]; moreover, there exists a nontrivial vector $\Omega \in \mathcal{H}$ (unique up to a phase) — called the vacuum vector — which is left invariant by $U(x,\Lambda)$:

$$U(x,\Lambda)\Omega = \Omega \ . \tag{9}$$

6. Additivity: For a covering $\mathcal{O}_\lambda$ of $\mathcal{O} \in \mathcal{K}$, $\mathfrak{A}(\mathcal{O})$ is contained in the von Neumann algebra generated by $\{\mathfrak{A}(\mathcal{O}_\lambda)\}$:

$$\mathcal{O} \subset \mathcal{O}_\lambda \Longrightarrow \mathfrak{A}(\mathcal{O}) \subset \bigvee \mathfrak{A}(\mathcal{O}_\lambda) \ . \tag{10}$$

Besides the net $\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O})$ [3] one has to specify the states viewed to be physically realizable or, equivalently, the representations obtained by GNS construction. Originally, Borchers investigated all positive energy representations, but it soon turned out that insurmountable difficulties arise from long range forces and infrared clouds in case the mass spectrum has no gap. Doplicher, Haag and Roberts therefore proposed the following criterion:

DHR criterion: *Consider those representations π which are unitarily equivalent to the vacuum representation π_0 when restricted to the causal complement of a sufficiently large double cone, i.e.*

$$\pi|_{\mathfrak{A}(\mathcal{O}')} \cong \pi_0|_{\mathfrak{A}(\mathcal{O}')} \tag{11}$$

for sufficiently large $\mathcal{O} \in \mathcal{K}$.

Remark: Unfortunately, "topological charges" (e.g. gauge charges) are ruled out by this criterion [4]. In case of theories with mass gap the situation has been improved by the analysis of Buchholz and Fredenhagen [3], who considered states localized in spacelike cones.

Nevertheless, we confine to states satisfying the DHR criterion in the present paper. In this case there is a unitary $V : \mathcal{H}_\pi \longrightarrow \mathcal{H}_0$ such that

$$\pi(A) = V\pi_0(A)V^* \ , \qquad A \in \mathfrak{A}(\mathcal{O}') \ . \tag{12}$$

In what follows we usually identify the quasilocal algebra $\mathfrak{A}$ with its realization in the vacuum Hilbert space, i.e. $A = \pi_0(A)$. Putting

$$\varrho(A) := V^*AV \ , \qquad A \in \mathfrak{A} \ , \tag{13}$$

[2] This condition is also called "positivity of energy".

[3] Note that the local algebras are in general hyperfinite factors of type III_1 — hence identical up to isomorphism; therefore, all physical information is contained in the *attachment* of the algebras $\mathfrak{A}(\mathcal{O})$ to the double cones!

[4] Think of electric charge: By Gauß' law the charge inside an arbitrarily large sphere can be determined by virtue of a flux measurement.

one realizes that the representation π describes a localized charge exactly if it is unitarily equivalent to a representation $\pi_0 \circ \varrho$, where ϱ is an *endomorphism of* $\mathfrak{A}$ *localized in* $\mathcal{O}$, i.e.

$$\varrho(A) = A \qquad \forall A \in \mathfrak{A}(\mathcal{O}') \ . \tag{14}$$

The crucial point is the possibility of composing endomorphisms; this way Doplicher, Haag and Roberts succeeded in analyzing the superselection structure [5].

Two localized morphisms ϱ_1, ϱ_2 are called *equivalent* if the representations $\varrho_1(\mathfrak{A})$ and $\varrho_2(\mathfrak{A})$ are unitarily equivalent; mutatis mutandis one defines irreducibility. A morphism ϱ localized in $\mathcal{O}$ is *transportable* if to each region $\tilde{\mathcal{O}}$ obtained from $\mathcal{O}$ by virtue of a Poincaré transformation there exists a morphism $\tilde{\varrho}$ localized in $\tilde{\mathcal{O}}$ equivalent to ϱ. Finally, a unitary $U \in \mathfrak{A}_{\rm loc}$ induces a localized, transportable automorphism of $\mathfrak{A}$ by virtue of $\sigma_U(A) := UAU^*$ — the so-called *inner automorphisms.*

By definition, the observables map each sector into itself; in case of several superselection sectors one is therefore interested in constructing additional *field operators* being local relative to the observables and inducing transitions between the different sectors. Following [6], the *field algebra* $\mathfrak{F}$ is obtained from $\mathfrak{A}$ by adjoining localized morphisms ϱ of $\mathfrak{A}$.

For permutation group statistics, Doplicher and Roberts were able — given the quasilocal algebra $\mathfrak{A}$ and the superselection structure — to recover a field algebra $\mathfrak{F}$ and a global gauge group $\mathfrak{G}$ such that $\mathfrak{A}$ is the $\mathfrak{G}$-invariant part of $\mathfrak{F}$, $\mathfrak{A} = \mathfrak{F}^{\mathfrak{G}}$ [9]. In the low dimensional case ($d \leq 2$, d the spatial dimension) braid group statistics may occur and the situation has not yet been completely clarified, but see [22], [26], [32].

According to the general theory of superselection sectors, for each ϱ localized in a double cone $\mathcal{O}$ there exists a unitary $\epsilon_\varrho \in \mathfrak{A}(\mathcal{O})$ commuting with $\varrho^2(\mathfrak{A})$ and fulfilling

$$\epsilon_\varrho \varrho(\epsilon_\varrho) \epsilon_\varrho = \varrho(\epsilon_\varrho) \epsilon_\varrho \varrho(\epsilon_\varrho) \ . \tag{15}$$

This so-called *statistics operator* is given by

$$\epsilon_\varrho = U^* \varrho(U) \ , \tag{16}$$

where U is a unitary such that

$$\tilde{\varrho}(A) := U \varrho(A) U^* \ , \qquad A \in \mathfrak{A} \ , \tag{17}$$

[5]The physical Hilbert space $\mathcal{H}$ decomposes into a direct sum of subspaces under the action of $\mathfrak{A}$; these (or rather the associated equivalence classes of unitary irreducible representations of $\mathfrak{A}$) are called superselection sectors.

is localized in a double cone $\tilde{\mathcal{O}} \subset \mathcal{O}'$. One easily checks that a unitary representation of the braid group B_n [6] is obtained by virtue of [11], [25], [27]

$$\epsilon_\varrho^{(n)}(\sigma_i) := \varrho^{i-1}(\epsilon_\varrho) \ , \qquad i = 1, \ldots, n-1 \ . \tag{18}$$

For $\sigma_i^2 = \mathbf{1}$, i.e. $\epsilon_\varrho^2 = \mathbf{1}$, one recovers permutation group statistics. Obviously, a localized, transportable morphism ϱ is in general not invertible, but there exists a *left inverse*, compare e.g. [28], i.e. a positive linear map $\phi_\varrho : \mathfrak{A} \longrightarrow \mathfrak{B}(\mathcal{H}_0)$ with

$$\begin{aligned} \text{(i)} \quad \phi_\varrho(A\varrho(B)) &= \phi_\varrho(A)B \ , \qquad A, B \in \mathfrak{A} \ ; & (19)\\ \text{(ii)} \quad \phi_\varrho(\mathbf{1}) &= \mathbf{1} \ . & (20) \end{aligned}$$

In case of an irreducible morphism, $\phi_\varrho(\epsilon_\varrho)$ is a multiple of the identity [17]:

$$\phi_\varrho(\epsilon_\varrho) = \lambda_\varrho \mathbf{1} \ ; \tag{21}$$

λ_ϱ is called the *statistical parameter* and may be written as $\lambda_\varrho = \frac{\omega_\varrho}{d_\varrho}$, $|\omega_\varrho| = 1$, $d_\varrho \geq 1$ (ω_ϱ is the *statistical phase* and d_ϱ the *statistical dimension* [7]). We will consider infra the case of ϱ being an automorphism; then ϵ_ϱ itself is a multiple of the identity, the induced representation of B_∞ is one-dimensional ($d_\varrho = 1$) and ω_ϱ is an arbitrary phase factor.

Exotic statistics has already been investigated by Streater and Wilde [36] and later by Wilczek [38], who introduced the term "anyon" for $d_\varrho = 1$ while for $d_\varrho > 1$ Fredenhagen, Rehren and Schroer suggested the term "plekton" [8] in [16].

2 Specification of fields and observables

Let H be a complex Hilbert space, $\mathcal{C}_0(H)$ the Clifford algebra over H and denote by $\mathcal{C}(H)$ the C^*-norm completion of $\mathcal{C}_0(H)$. Furthermore, let $\Psi : H \longrightarrow \mathcal{C}(H)$ be an antilinear injection fulfilling canonical anticommutation relations:

$$[\Psi(f), \Psi(g)]_+ = 0 \quad , \qquad [\Psi(f), \Psi(g)^*)]_+ = \langle f, g \rangle \ \mathbf{1} \ . \tag{22}$$

[6] B_n is generated by the identity and elements σ_i, $i = 1, \ldots, n-1$, fulfilling the *Artin relations*

$$\begin{aligned} \sigma_i \sigma_j &= \sigma_j \sigma_i \quad |i-j| \geq 2 \ ; \\ \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1} \ . \end{aligned}$$

B_∞ is the inductive limit of the B_n with respect to the canonical inclusions.

[7] d_ϱ is related to the index of the inclusion $\varrho(\mathfrak{A}(\mathcal{O})) \subset \mathfrak{A}(\mathcal{O})$ by means of $\mathrm{Ind}\varrho = d_\varrho^2$ [21].

[8] The Greek expression for "braided".

Next we introduce a family of automorphisms of $\mathcal{C}(H)$ important for our purposes. If

$$U(H) := \{U \in \mathfrak{B}(H) \mid UU^* = U^*U = \mathbf{1}\} \tag{23}$$

denotes the group of unitary Bogoliubov operators, then every $U \in U(H)$ induces an automorphism of $\mathcal{C}(H)$ by

$$\alpha_U(\Psi(f)) = \Psi(Uf) \; . \tag{24}$$

In addition, for $p \geq 1$ we denote by $\mathcal{J}_p$ the trace ideal $\{A \in \mathfrak{B}(H) \mid \mathrm{Tr}\, |A|^p < \infty\}$.

For every projection P in H , there is a unique pure quasifree [9] gauge invariant state ω_P on $\mathcal{C}(H)$ with two point function

$$\omega_P(\Psi(f)\Psi(g)^*) = \langle f, Pg\rangle \; , \tag{26}$$

i.e. a Fock state; let $(\mathcal{H}_P, \pi_P, \Omega_P)$ denote the associated GNS triple. It is well known [34] that an automorphism α_U, $U \in U(H)$, is unitarily implementable in a Fock representation π_P, i.e. there exists $\Gamma(U) \in U(\mathcal{H}_P)$ with $\pi_P(\Psi(Uf)) = \Gamma(U)\pi_P(\Psi(f))\Gamma(U)^*$, if and only if $PU(\mathbf{1}-P)$ and $(\mathbf{1}-P)UP$ are Hilbert-Schmidt operators.

Let us now specify the fields and observables necessary for our considerations. Put $H := L^2(\mathbb{R}, \mathbb{C}^2)$ and choose a polarization on H according to the positive and negative part of the spectrum of D_m, the Dirac operator of mass $m > 0$; denote the corresponding projections by P_+ resp. P_-. Moreover, let α_t be the automorphism generated by $\mathrm{e}^{\mathrm{i}tD_m}$ and $T(x)$ the operator on H representing a translation by x. Denoting by $\overline{}$ the conjugate Hilbert space, the one particle space is given by $\mathcal{H}_1 := P_+H \oplus \overline{P_-H}$ and the physical Hilbert space $\mathcal{H}$ is precisely the antisymmetric Fock space over $\mathcal{H}_1$:

$$\mathcal{H} = \mathcal{F}_a(\mathcal{H}_1) \; . \tag{27}$$

For a denoting the annihilation operator, we thus obtain a positive energy representation π of $\mathcal{C}(H)$ on $\mathcal{H}$ by virtue of

$$\pi(\Psi(f)) = a(P_+f) + a^*(\overline{P_-f}) \; , \tag{28}$$

this being the only positive energy representation of $\mathcal{C}(H)$ with respect to the dynamics given by $(\alpha_t)_{t\in\mathbb{R}}$ in the massive case [37]. Let

$$U_2(H) := \{U \in U(H) \mid P_\pm U P_\mp \in \mathcal{J}_2(H)\} \tag{29}$$

[9] Recall that a state on $\mathcal{C}(H)$ is called quasifree if its n-point functions are given by

$$\omega_A(\Psi(f_n)\ldots\Psi(f_1)\Psi(g_1)^*\ldots\Psi(g_m)^*) = \delta_{mn}\,\det\left((f_i, Ag_j)\right) \; , \tag{25}$$

where A is an operator on H satisfying $0 \leq A \leq \mathbf{1}$.

be the group of Bogoliubov operators unitarily implementable in π and $\mathfrak{u}_2(H)$ its complex Lie algebra. If $X \in \mathfrak{u}_2(H)$ is self-adjoint, $\mathrm{e}^{\mathrm{i}tX}$ is a norm-continuous one-parameter subgroup of $U_2(H)$ and we choose the phases in the corresponding implementers such that we get a strongly continuous one-parameter group, i.e. $\mathrm{d}\Gamma(X)$ being the self-adjoint generator, we have

$$\Gamma(\mathrm{e}^{\mathrm{i}tX}) = \mathrm{e}^{\mathrm{i}t\mathrm{d}\Gamma(X)} \qquad \forall X = X^* \in \mathfrak{u}_2(H) \; . \tag{30}$$

The additive constant in $\mathrm{d}\Gamma(X)$ is fixed by demanding its vacuum expectation value to vanish [10].

Remark: The reader will have noticed that our formalism to quantize relativistic fermions is equivalent to a more geometric approach discussed in detail in [23]. Roughly speaking, those vectors in Fock space corresponding to pure quasifree states are identified with sections in a complex line bundle $\mathbf{DET}^*$ (the dual determinant bundle) over a complex Hilbert manifold $\mathbf{Gr}(H)$, the infinite dimensional Grassmannian over H. More precisely, for a Hilbert space $H = H_+ \oplus H_-$ with given polarization one defines the Grassmannian $\mathbf{Gr}(H)$ to be the set of closed subspaces W of H such that

(i) $P_+ : W \longrightarrow H_+$ is a Fredholm operator;

(ii) $P_- : W \longrightarrow H_-$ is a Hilbert-Schmidt operator.

$\mathbf{Gr}(H)$ is a smooth complex Hilbert manifold modeled on the space $\mathcal{J}_2(H_+, H_-)$ of Hilbert-Schmidt operators from H_+ to H_-; its connected components are labeled by the Fredholm index of P_+. The group $GL(H)$ of bounded invertible operators on H does not act on $\mathbf{Gr}(H)$; therefore, one passes to the so-called restricted linear group $GL_2(H)$, which is the complexification of the real Banach Lie group $U_2(H)$. The action of $U_2(H)$ on $\mathbf{Gr}(H)$ extends to a holomorphic action of a central extension $U_2(H)^\sim$ on $\mathbf{DET}^*$, i.e. one obtains a representation of $U_2(H)^\sim$ on Fock space.

Based on previous results of Fredenhagen [10] and Klaus and Scharf [20], Carey, Hurst and O'Brien [4] observed that if one considers the splitting

$$\mathcal{H} = \bigoplus_{q\in\mathbb{Z}} \mathcal{H}_q \tag{31}$$

of the physical Hilbert space into charge sectors, the implementer $\Gamma(U)$ of $U \in U_2(H)$ [11] maps $\mathcal{H}_q$ onto $\mathcal{H}_{q+q(U)}$, where $q(U)$ is the Fredholm index of P_-UP_- [12]. In more detail, they found that if $\Gamma(U)$ acts on a state Φ of p particles and q antiparticles, these numbers are changed to $p + \dim\ker P_-UP_-$ and $q + \dim\ker P_+UP_+$.

For a double cone $\mathcal{O}$ with basis $B_\mathcal{O}$ at time t we define

$$\mathfrak{F}(\mathcal{O}) := \{\pi(\alpha_t(\Psi(f))) \mid \mathrm{supp}\, f \in B_\mathcal{O}\}'' \tag{32}$$

to be the associated local field algebra; put $\mathfrak{F} := \overline{\bigcup_{\mathcal{O}\in\mathcal{K}} \mathfrak{F}(\mathcal{O})}^{\|\,\|_{C^*}}$. Since Fermi fields anticommute rather than commute if localized in space-like

[10] An argument that this can be done is given e.g. in [5].

[11] This is a somewhat sloppy manner of speaking, of course; however, we will not distinguish between U and the automorphism α_U induced by it.

[12] Clearly, P_+UP_+ and P_-UP_- are Fredholm operators for any $U \in U_2(H)$.

separated regions, Haag duality is not the appropriate notion of duality but has to be replaced by what is called *twisted duality* : For $F \in \mathfrak{F}$ one defines the bosonic resp. fermionic part of F by

$$F_{\pm} := \frac{1}{2}(F \pm \mathrm{Ad}\,\Gamma(-\mathbf{1})(F)) \tag{33}$$

and sets

$$\mathfrak{F}(\mathcal{O})^{\tau} := \{F_{+} + \Gamma(-\mathbf{1})F_{-} \mid F \in \mathfrak{F}(\mathcal{O})\} . \tag{34}$$

Then, instead of Haag duality, one has [6]

$$\mathfrak{F}(\mathcal{O})^{\tau} = \mathfrak{F}(\mathcal{O}')' . \tag{35}$$

The local observable algebras consist of the $U(1)$-invariant elements of the local field algebras:

$$\mathfrak{A}(\mathcal{O}) := \mathfrak{F}(\mathcal{O}) \cap \{\Gamma(\mathrm{e}^{\mathrm{i}\gamma}) \mid \gamma \in \mathbb{R}\}' . \tag{36}$$

Note that $\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O})$ is just the net of observables of the free Dirac field in two space-time dimensions. Unfortunately — as we mentioned before — this net does not fulfil Haag duality: Without loss of generality we may consider the situation at time $t = 0$; if $\mathcal{O}, \mathcal{O}_1, \mathcal{O}_2$ are double cones such that $\mathcal{O}_1$ and $\mathcal{O}_2$ lie in different connected components of $\mathcal{O}'$ and f_1 resp. f_2 are test functions with support in $\mathcal{O}_1$ resp. $\mathcal{O}_2$, then $\pi(\Psi(f_1))^*\pi(\Psi(f_2))$ is contained in $\mathfrak{A}(\mathcal{O})'$ but not in $\mathfrak{A}(\mathcal{O}')''$.

3 Construction of automorphisms

Generalizing the construction of Binnenhei [2], we will now obtain localized automorphisms showing (abelian) braid group statistics of the quasilocal algebra $\mathfrak{A}$ associated with the net of observables defined in the preceding section.

Before going into details, let us briefly outline our strategy. To begin with, $\pi(\mathcal{C}(H))$ is contained in $\mathfrak{F}$ as a weakly dense subalgebra and $\pi(\mathcal{C}(H)^{U(1)})$ [13] is weakly dense in $\mathfrak{A}$. Therefore, it is natural to study automorphisms α of $\mathcal{C}(H)$ which leave $\mathcal{C}(H)^{U(1)}$ invariant and extend to automorphisms of $\mathfrak{A}$; moreover, the extension of $\alpha|_{\mathcal{C}(H)^{U(1)}}$ should be localized in some double cone $\mathcal{O}$ and transportable. Finally, as the restriction of π to $\mathcal{C}(H)^{U(1)}$ decomposes into a direct sum of mutually inequivalent irreducible representations (see above),

$$\pi|_{\mathcal{C}(H)^{U(1)}} = \bigoplus_{q\in\mathbb{Z}} \pi_q \text{ [14]} , \tag{37}$$

[13] $\mathcal{C}(H)^{U(1)}$ denotes the $U(1)$-invariant part of $\mathcal{C}(H)$, of course.

[14] π_q the restriction of $\pi|_{\mathcal{C}(H)^{U(1)}}$ to $\mathcal{H}_q$

we look for automorphisms connecting the vacuum to the charged sectors, i.e.

$$\pi_0 \circ \alpha|_{\mathcal{C}(H)^{U(1)}} \cong \pi_q \tag{38}$$

for some $q \in \mathbb{Z}^*$. It is known [2] that all these conditions are met for automorphisms induced by a certain class of Bogoliubov operators $U \in U_2(H)$, and our morphisms are precisely of this form in the sense that

$$\alpha_U(A) = \Gamma(U) A \Gamma(U)^* \; , \qquad A \in \mathfrak{A} \; . \tag{39}$$

To this end, for $\varepsilon > 0$ let h_ε be an odd, real-valued, smooth function on $\mathbb{R}$ being equal to 1 for $x \geq \varepsilon$ and increasing monotonously inside the interval $(-\varepsilon, \varepsilon)$. Define unitary multiplication operators on H by

$$(U(n,\lambda)f)(x) := \begin{pmatrix} \mathrm{e}^{\mathrm{i}\pi(n+\lambda)h_\varepsilon(x)} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\pi\lambda h_\varepsilon(x)} \end{pmatrix} f(x) \; , \qquad n \in \mathbb{Z} \; , \; \lambda \in \mathbb{R} \; , \tag{40}$$

and

$$(A(\lambda)f)(x) := \begin{pmatrix} \pi\lambda h_\varepsilon(x) & 0 \\ 0 & \pi\lambda h_\varepsilon(x) \end{pmatrix} f(x) \; , \qquad \lambda \in \mathbb{R} \; , \tag{41}$$

i.e. $U(0,\lambda) = \mathrm{e}^{\mathrm{i}A(\lambda)}$. In what follows we will write just U instead of $U(n,\lambda)$ whenever no confusion can arise.

Lemma 1 *$P_\pm U P_\mp \in \mathcal{J}_2(H) \quad \forall n \in \mathbb{Z}, \forall \lambda \in \mathbb{R}$ and $m > 0$, i.e. the automorphism α_U of $\mathcal{C}(H)$ induced by U is unitarily implementable in the Fock representation π for all integers n and all real λ in the massive case.*

Proof: Straightforward calculation; we refer to [4], [5], [30] for detailed proofs of this statement. □

Remark: For $m = 0$ the operators $P_\pm U P_\mp$ are Hilbert-Schmidt only for $\lambda \in \mathbb{Z}$ [24], [30]. As we shall see below, this case does not yield interesting morphisms in the sense that braid group statistics cannot occur; hence we confine to the massive case.

Lemma 2 *$q(U) = n$.*

Proof: This follows immediately from the index formulae for generalized Wiener-Hopf operators proven by Ruijsenaars in [31]. Namely, he has shown that the index of $P_- U(n,0) P_-$ equals the winding number of $\mathrm{e}^{\mathrm{i}\pi n h_\varepsilon(\cdot)}$ [15]; by continuity in λ, $P_- U(0,\lambda) P_-$ lies in the connected component of the identity and has therefore vanishing index. □

[15] By convention, we choose the winding number of

$$x \mapsto \frac{x - i}{x + i}$$

to be positive.

Proposition 1 *U induces a localized, transportable automorphism α_U of $\mathfrak{A}$ by means of $\alpha_U(A) = \Gamma(U)A\Gamma(U)^*$.*

Proof: For simplicity, we may restrict to the situation at $t = 0$. By unitarity of U, α_U maps $\mathcal{C}(H)^{U(1)}$ into itself; moreover, the extension of $\alpha_U|_{\mathcal{C}(H)^{U(1)}}$ to $\mathfrak{A}$ causes no problem.

Concerning localizability, an operator $V \in U_2$ certainly does induce a morphism localized in $\mathcal{O}$ if for each connected component $\Delta_i, i = 1, 2,$ of $\mathcal{O}'$ there exists $\tau_i \in U(1)$ such that $Vf = \tau_i f$ for all $f \in H$ with supp $f \in \Delta_i$ [16], and U obviously shares this property.

Now, if U induces a morphism localized in $\mathcal{O}$, $U_x := T(x)UT(-x)$ generates one localized in $(\mathcal{O}+x)$. We claim that these morphisms are equivalent by means of a unitary element of a local observable algebra. To see this, consider the operator $\Gamma(U_x U^*)$. Clearly, $\Gamma(U_x U^*)$ maps the charge sectors $\mathcal{H}_q$ into themselves and commutes with global gauge transformations $\Gamma(\mathrm{e}^{\mathrm{i}\gamma})$. Let $\tilde{\mathcal{O}} \supset \mathcal{O} \cup (\mathcal{O} + x)$; then $\Gamma(U_x U^*)$ is contained in $\mathfrak{A}(\tilde{\mathcal{O}})$: it remains to be shown that $\Gamma(U_x U^*) \in \mathfrak{F}(\tilde{\mathcal{O}})^\tau = \mathfrak{F}(\tilde{\mathcal{O}}')'$; choosing $f \in H$ such that supp $f \cap B_{\tilde{\mathcal{O}}} = \emptyset$, we have

$$\begin{aligned} \Gamma(U_x U^*)\pi(\Psi(f))\Gamma(U_x U^*)^* &= \pi(\Psi(U_x U^* f)) \\ &= \pi(\Psi(f)) \ . \end{aligned}$$

□

Remark: The fact that the charge transfer operators lie in a local observable algebra is not clear a priori since this is usually shown using Haag duality — which is violated here.

To summarize, we succeeded in constructing localized, transportable automorphisms of $\mathfrak{A}$ connecting the vacuum to the different charged sectors. Our next aim is to compute the corresponding statistics operator ϵ_{α_U}. To do this, we need two more lemmata.

Lemma 3 **(i)** *Let $X, Y \in \mathfrak{u}_2(H)$ be self-adjoint operators such that $[X, Y] = 0$. Then*

$$\Gamma(\mathrm{e}^{\mathrm{i}X})\Gamma(\mathrm{e}^{\mathrm{i}Y}) = \mathrm{e}^{-c(X,Y)}\Gamma(\mathrm{e}^{\mathrm{i}Y})\Gamma(\mathrm{e}^{\mathrm{i}X}) \ , \tag{42}$$

where $c(X, Y) := \mathrm{Tr}\ (P_- X P_+ Y P_- - P_- Y P_+ X P_-)$.

(ii) *For $V \in U_2(H)$ and $X = X^* \in \mathfrak{u}_2(H)$ with $[V, X] = 0$,*

$$\Gamma(\mathrm{e}^{\mathrm{i}X})\Gamma(V) = \mathrm{e}^{\mathrm{i}(\Gamma(V)\Omega, \mathrm{d}\Gamma(X)\Gamma(V)\Omega)}\Gamma(V)\Gamma(\mathrm{e}^{\mathrm{i}X}) \tag{43}$$

holds true.

[16] In fact, this condition is both necessary and sufficient [2].

Proof: That Γ is a projective representation of $U_2(H)$ with (Lie group) cocycle $C(X,Y) = \mathrm{e}^{-c(X,Y)}$ is well known, of course [5], [23]; (ii) follows by uniqueness of the implementers [5]. □

Remark: Actually, $c(X,Y)$ is the (Lie algebra) cocycle associated with $\mathrm{d}\Gamma$ fixing the central extension and known to physicists as the *Schwinger term* occuring in the current algebra.

In addition, denoting by $\mathcal{O}_x$ the double cone with basis $(x-\varepsilon, x+\varepsilon)$ at time $t=0$, we obtain

Lemma 4 *For $\mathcal{O}_x$ space-like to $\mathcal{O}_{x'}$, one has*

(i)

$$\Gamma(U_x(n,0))\Gamma(U_{x'}(n',0)) = (-1)^{nn'}\Gamma(U_{x'}(n',0))\Gamma(U_x(n,0)) \; ; \qquad (44)$$

(ii)

$$c(A_x(\lambda), A_{x'}(\lambda')) = 0 \; ; \qquad (45)$$

[17]

(iii)

$$(\Gamma(U_x(n,0))\Omega, \mathrm{d}\Gamma(A_{x'}(0,\lambda))\Gamma(U_x(n,0))\Omega) = \pi n\lambda \mathrm{sign}\ (x-x') \; . \quad (46)$$

Proof: Again, (i) is well known [23]; to prove (ii), notice that the relevant convolution kernels are given by

$$P_\delta A P_{-\delta}(\theta,\theta') = N\delta\hat{h}_\varepsilon[\delta(\sinh\theta + \sinh\theta')]\sinh\left(\frac{\theta+\theta'}{2}\right) , \qquad \delta = +,- \; ,$$

where $\hat{h}_\varepsilon$ is the (distributional) Fourier transformation of h_ε, θ denotes the rapidity (recall that the momentum p is related to the rapidity by $p(\theta) = m\sinh\theta$) and N is some constant; since h_ε is an odd function, we see that $P_+AP_- = P_-AP_+$. (iii) is obvious. □

Now we are in a position to state

Proposition 2 $\epsilon_{\alpha_U} = (-1)^n \mathrm{e}^{\mathrm{i}2\pi n\lambda \mathrm{sign}\ (x'-x)}\mathbf{1}$.

Proof: By definition, we have

$$\epsilon_{\alpha_U} = \Gamma(U_x U^*_{x'})^*\Gamma(U_x)\Gamma(U_x U^*_{x'})\Gamma(U_x)^* \; .$$

Writing $\Gamma(U_x)$ as $C(U_x(n,0), U_x(0,\lambda))^{-1}\Gamma(U_x(n,0))\Gamma(U_x(0,\lambda))$ [18], the statement follows immediately from lemmata 3 and 4. □

Remark: Consider the case of automorphisms α_U resp. $\alpha_{U'}$ induced by Bogoliubov operators $U = U(n,\lambda)$ resp. $U' = U(n',\lambda')$; using the fact that localized, transportable morphisms commute if localized in space-like separated regions (see e.g. [28]), the statistics operator is again a multiple of the identity; namely, we obtain

$$\epsilon(\alpha_U, \alpha_{U'}) = (-1)^{nn'}\mathrm{e}^{\mathrm{i}\pi(n\lambda'+n'\lambda)\mathrm{sign}\ (x'-x)}\mathbf{1} \; . \qquad (47)$$

[17] $A_x := T(x)AT(-x)$

[18] $C(.,.)$ denoting again the (Lie group) cocycle corresponding to the projective representation Γ.

4 Extension of $\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O})$

In this section we extend the net $\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O})$ to obtain one fulfilling Haag duality and investigate the superselection structure of the latter.

Remark: Since $\Gamma(U_x(0,\lambda))$ is contained in $\mathfrak{A}(\mathcal{O}_x')'$ but not in $\mathfrak{A}(\mathcal{O}_x)$, one way to extend the net of observables is to adjoin $\Gamma(U_x(0,\lambda))$ to $\mathfrak{A}(\mathcal{O}_x)$ and consider the von Neumann algebra such generated as the new local observable algebra; this is equivalent to taking the dual net $\mathfrak{A}^d(\mathcal{O}) := \mathfrak{A}(\mathcal{O}')'$. Unfortunately, the operators $\Gamma(U_x(0,\lambda))$ are localized only in half-spaces (after performing a global gauge transformation), and one has to study algebras $\mathfrak{A}(W_\pm)$ associated with wedge regions

$$W_\pm := \{x \in \mathbb{R} \mid |x^0| < \pm x^1\} . \tag{48}$$

For these algebras Haag duality is a consequence of twisted duality [6]:

$$\mathfrak{A}(W_+ + x)' = \mathfrak{A}(W_- + x)'' . \tag{49}$$

In this case the natural extension of α_U may be interpreted as a soliton [19].

The extension we want to study is analogous to the universal algebra introduced by Fredenhagen, Rehren and Schroer in the context of conformal field theory [16]. Namely, we define

$$\begin{aligned} \mathfrak{A}_1(\mathcal{O}) &:= \mathfrak{A}(\mathcal{O}) , & (50)\\ \mathfrak{A}_1(\mathcal{O}') &:= \mathfrak{A}(\mathcal{O})' . & (51)\end{aligned}$$

By construction, this net [20] fulfils Haag duality. The algebras $\mathfrak{A}_1(\mathcal{O})$ and $\mathfrak{A}_1(\mathcal{O}')$ generate a C^*-algebra $\mathfrak{A}_1$ uniquely determined by the following properties:

(i) There exist unital embeddings $\imath^I : \mathfrak{A}_1(I) \hookrightarrow \mathfrak{A}_1$ such that for all $I, J \in \Sigma := \{\mathcal{O}, \mathcal{O}' \mid \mathcal{O} \in \mathcal{K}\}$ one has

$$\imath^J|_{\mathfrak{A}_1(I)} = \imath^I , \qquad I \subset J , \tag{52}$$

and $\mathfrak{A}_1$ is generated by the algebras $\imath(\mathfrak{A}_1(I))$.

(ii) For every family $(\pi^I)_{I\in\Sigma}$, $\pi^I : \mathfrak{A}_1(I) \longrightarrow \mathfrak{B}(\mathcal{H}_\pi)$, of normal representations with

$$\pi^J|_{\mathfrak{A}_1(I)} = \pi^I , \qquad I \subset J , \tag{53}$$

there is a unique representation π of $\mathfrak{A}_1$ in $\mathcal{H}_\pi$ such that

$$\pi \circ \imath^I = \pi^I . \tag{54}$$

[19]Note that in our case the different vacua are connected by an internal symmetry; for a generalization we refer to [13].

[20]To be honest, the notion of a net is used only for directed index sets; what we have defined is rather a precosheaf. However, we ignore such subtleties and keep on using the term net.

Note that the vacuum representation π_0 of $\mathfrak{A}_1$ induced by $\imath^I(A) \mapsto A$ is in general not faithful.

Our next task is to extend α_U to $\mathfrak{A}_1$. For notational simplicity, let us consider the situation at $t = 0$. Since we already know how α_U operates on local algebras $\mathfrak{A}_1(\mathcal{O})$, only its action on $\mathfrak{A}_1(\mathcal{O}')$, i.e. on the commutants $\mathfrak{A}(\mathcal{O})'$, remains to be determined. Let α_U be localized in $\mathcal{O}$ and let $\mathcal{O}_x$ [21] be another double cone. Then α_U is unitarily equivalent to a morphism localized in $\mathcal{O}_x$ by means of

$$\mathfrak{T} := \Gamma(U_x U^*) \ . \tag{55}$$

As in the proof of proposition 1, we see that this charge transfer operator is contained in a local algebra $\mathfrak{A}_1(\tilde{\mathcal{O}})$, $\tilde{\mathcal{O}} \supset \mathcal{O} \cup \mathcal{O}_x$. We define

$$\alpha_U^{\mathfrak{A}_1(\mathcal{O}'_x)} := \mathrm{Ad}\ \mathfrak{T}^*|_{\mathfrak{A}_1(\mathcal{O}'_x)} \ . \tag{56}$$

We claim that this is well-defined: For if A is in $\mathfrak{A}_1(\mathcal{O}'_{x_1})$ as well as in $\mathfrak{A}_1(\mathcal{O}'_{x_2})$, both $\mathrm{Ad}\ \Gamma(U_{x_1})$ and $\mathrm{Ad}\ \Gamma(U_{x_2})$ act trivially on A; therefore,

$$\alpha_U^{\mathfrak{A}_1(\mathcal{O}'_{x_1})}(A) = \alpha_U^{\mathfrak{A}_1(\mathcal{O}'_{x_2})}(A) \ . \tag{57}$$

Theorem 1 *The extensions to $\mathfrak{A}_1$ of morphisms α_U resp. $\alpha_{U'}$ induced by Bogoliubov operators $U = U(n, \lambda)$ resp. $U' = U(n', \lambda')$ are inequivalent provided that $n \neq n'$ or $\lambda \neq \lambda'$ $(\mathrm{mod}\ 2\mathbb{Z})$.*

Proof: Obviously, α_U and $\alpha_{U'}$ cannot be equivalent if $n \neq n'$, as in this case they connect the vacuum to different charge sectors. Moreover, for a net fulfilling Haag duality, the notions of unitary equivalence and inner equivalence coincide: For there exists a double cone $\mathcal{O}$ containing the supports both of α_U and $\alpha_{U'}$, and the morphisms act trivially on $\mathfrak{A}_1(\mathcal{O}')$; but then the unitary operator providing the equivalence commutes with $\mathfrak{A}_1(\mathcal{O}')$ and therefore is in $\mathfrak{A}_1(\mathcal{O})$. Now, according to the general theory of superselection sectors, if α_U and $\alpha_{U'}$ were inner equivalent, their statistics operators had to coincide. □

Since we did not enlarge the local algebras themselves but only redefined the algebras associated with the (unbounded) causal complements of double cones, the additional quantum number λ is of a topological type and might be thought of as a "charge at infinity".

[21] Note that the restriction to double cones with basis $(x - \varepsilon, x + \varepsilon)$ does not mean any loss of generality.

5 Scattering theory

In this final section we assume $n = 1$. As we mentioned in section 1, the theorem of Doplicher and Roberts, yielding a field algebra $\mathfrak{F}$ once the quasilocal algebra $\mathfrak{A}$ and the superselection structure are known, does not apply to our case; therefore, we have to use a substitute — the so-called *field bundle* $\mathcal{F}$ [7] — to describe charged fields interpolating between different sectors. In this framework, state vectors are pairs $\{\varrho, \phi\}$, $\phi \in \mathcal{H}$ and ϱ a localized morphism of $\mathfrak{A}$. Furthermore, fields are pairs $\{\varrho, A\}$, $A \in \mathfrak{A}$, and act on state vectors by virtue of

$$\{\varrho, A\}\{\varrho', \phi\} = \{\varrho'\varrho, \varrho'(A)\phi\} . \tag{58}$$

A field $\{\varrho, A\}$ is localized in a double cone $\mathcal{O}$ if

$$AB = \varrho(B)A , \qquad B \in \mathfrak{A}(\mathcal{O}') . \tag{59}$$

Finally, for any two localized morphisms ϱ_1 and ϱ_2, one defines the set $\mathfrak{I}(\varrho_2|\varrho_1)$ of intertwiners from ϱ_1 to ϱ_2:

$$\mathfrak{I}(\varrho_2|\varrho_1) := \{T \in \mathfrak{B}(\mathcal{H}_0) \mid \varrho_2(A)T = T\varrho_1(A) \ \ \forall A \in \mathfrak{A}\} . \tag{60}$$

Remark: At this point, let us comment on a difficulty showing up if one considers the universal algebra for conformal field theories living on the circle, i.e. $\mathfrak{A}_1 = \mathfrak{A}_1(S^1)$, namely, the existence of global self-intertwiners [16]. The problem is that the statistics operator may depend on the choice of a "point at infinity" $\xi \in S^1$. For if ϱ is localized in an interval J, σ is localized in an interval I and $\xi \in J' \cap I'$, the statistics operator $\epsilon_\xi(\varrho, \sigma)$ does not change as long as ξ varies continuously; but $J' \cap I'$ may consist of two connected components Δ_1, Δ_2. Choosing $\xi \in \Delta_1$ and $\zeta \in \Delta_2$, the corresponding statistics operators coincide only if the monodromy is trivial!

Global self-intertwiners arise as follows: Let ϱ and σ be localized in I and $\tilde{\varrho}$ a morphism equivalent to ϱ localized in J; then $\epsilon(\varrho, \sigma)$ and $\epsilon(\sigma, \varrho)$ coincide for ξ and ζ. Putting

$$\mathfrak{A}_\xi := \overline{\{\mathfrak{A}(I) \mid I \in S^1 , \xi \notin I\}} , \tag{61}$$

both $\epsilon(\varrho, \sigma)$ and $\epsilon(\sigma, \varrho)$ are in $\mathfrak{A}_\xi \cap \mathfrak{A}_\zeta$; by Haag duality, there exists an intertwiner $S : \pi_0\varrho \longrightarrow \pi_0\tilde{\varrho}$ contained in $\pi_0(\mathfrak{A}_\xi)$ as well as in $\pi_0(\mathfrak{A}_\zeta)$. Denoting the corresponding preimages by $S_1 \in \mathfrak{A}_\xi$ resp. $S_2 \in \mathfrak{A}_\zeta$, a global self-intertwiner from ϱ to ϱ is given by

$$S_\varrho := S_1^* S_2 . \tag{62}$$

The crucial point is that S_ϱ is trivial in the vacuum representation while $\pi_0\sigma(S_\varrho)$ is the monodromy operator:

$$\pi_0\sigma(S_\varrho) = \pi_0(\epsilon(\varrho, \sigma)\epsilon(\sigma, \varrho)) . \tag{63}$$

Roughly speaking, the choice of a "point at infinity" determines whether I lies to the right of J or vice versa. For theories on the real line this difficulty is absent due to the fact that a "point at infinity" is given a priori.

For convenience of the reader not familiar with our approach, let us now sketch the Haag-Ruelle programme; details may be found e.g. in [7], [14].

Take $B' \in \mathfrak{A}_1(\mathcal{O})$ for some double cone $\mathcal{O}$ and let $\mathbf{B'} = \{\varrho, B'\}$ be the corresponding field. Moreover, let g be a Schwartz function on Minkowski space whose Fourier transform intersects the spectrum of $\mathbf{B'}\Omega$ only on the mass hyperboloid $p^2 = m^2$. Define a single particle state by virtue of

$$\mathbf{B} := \int \mathrm{d}^2 x g(x) \alpha_x(\mathbf{B'}) \ . \tag{64}$$

Our choice of g and B' guarantees $\mathbf{B}$ to be almost local; if

$$f(x) = \int \mathrm{d}p \hat{f}(p) \mathrm{e}^{\mathrm{i}(px^1 - E_p t)} \qquad (\hat{f} \in \mathcal{C}_0^\infty \ , \ \ E_p = \sqrt{p^2 + m^2}) \tag{65}$$

is a positive energy solution of the Klein-Gordon equation for mass m, one puts

$$\mathbf{B}_f(t) := \int_{x^0 = t} \mathrm{d}x^1 f(x) \alpha_x(\mathbf{B}) \tag{66}$$

and obtains an eigenvector of the mass operator M^2 with eigenvalue m^2 (i.e. a one particle state vector) by means of

$$\Psi := \mathbf{B}_f(t)\Omega = \{\varrho, \psi\} \ ; \tag{67}$$

Ψ does not depend on t. The next step is to define multiparticle states by means of

$$\mathbf{B}_{f_n}(t) \ldots \mathbf{B}_{f_1}(t)\Omega \ ; \tag{68}$$

for reasons of convergence one assumes the functions f_i to have mutually disjoint supports in rapidity space. Then one proves strong convergence of the expressions

$$\lim_{t \to -\infty} \mathbf{B}_{f_n}(t) \ldots \mathbf{B}_{f_1}(t)\Omega =: \Psi_n \times_{in} \ldots \times_{in} \Psi_1 \tag{69}$$

resp.

$$\lim_{t \to +\infty} \mathbf{B}_{f_n}(t) \ldots \mathbf{B}_{f_1}(t)\Omega =: \Psi_n \times_{out} \ldots \times_{out} \Psi_1 \tag{70}$$

and interpretes them as incoming resp. outgoing scattering states.

For later use we state without proof a version of the well known cluster theorem [14]:

Lemma 5 *Let $\mathbf{B}_i = \{\varrho_i, B_i\} \in \mathcal{F}(\mathcal{O}_i)$, $i = 1, \ldots, 4$, such that $\mathcal{O}_1 \cup \mathcal{O}_3 \times \mathcal{O}_2 \cup \mathcal{O}_4$. Furthermore, let T be an intertwiner from $\varrho_1\varrho_2$ to $\varrho_3\varrho_4$ and define*

$$\tau := \sup\{|t| \mid \mathcal{O}_1 \cup \mathcal{O}_3 \ + \ (t, 0) \subset (\mathcal{O}_2 \cup \mathcal{O}_4)'\} \ . \tag{71}$$

Assuming ϱ_4 to be irreducible with finite statistics (i.e. $d_{\varrho_4} < \infty$) and (unique) right inverse χ_4 [22] *and denoting by $\{W_j\}$ a (possibly empty) orthonormal basis of the Hilbert space of local intertwiners from ϱ_4 to ϱ_2, one has*

$$|(\mathbf{B}_4\mathbf{B}_3\Omega, T\mathbf{B}_2\mathbf{B}_1\Omega) - \sum_j (\mathbf{B}_3\Omega, \chi_4(T\varrho_1(W_j))\mathbf{B}_1\Omega)(\mathbf{B}_4\Omega, W_j^*\mathbf{B}_2\Omega)| \leq \mathrm{e}^{-\mu\tau} \prod_i \|\mathbf{B}_i\| \ . \tag{72}$$

To apply this programme to our morphisms α_U, we have to check that they induce massive single particle representations; but this is clear: Covariance is a consequence of covariance of the Bogoliubov operators U_x:

$$U_{x+a} = T(a)U_x T(-a) \ , \tag{73}$$

and the generators of translations coincide with those for the free field; hence there exists an isolated mass shell.

Let $\{\Psi_i\} = \{\mathbf{B}_{f_i}(t)\Omega\} = \{\{\varrho_i, \psi_i\}\}$, $i = 1, \dots n$, be a set of normed single particle states with mutually disjoint supports in rapidity space, where all ϱ_i are taken to be copies of α_U. Without loss of generality we assume the support of f_i in rapidity space left to that of f_{i+1}. Hence for $t \to -\infty$ we have

$$(\Psi_n, \mathcal{O}_n) < \dots < (\Psi_1, \mathcal{O}_1) \ ^{23} \ , \tag{74}$$

while for $t \to +\infty$ we obtain

$$(\Psi_1, \mathcal{O}_1) < \dots < (\Psi_n, \mathcal{O}_n) \ . \tag{75}$$

These incoming resp. outgoing configurations are linked by means of the unitary transformation

$$S(\lambda) = (-\mathrm{e}^{\mathrm{i}2\pi\lambda})^{\sum_{i<j} 1}\mathbf{1} = (-\mathrm{e}^{\mathrm{i}2\pi\lambda})^{\frac{n(n-1)}{2}}\mathbf{1} \ . \tag{76}$$

Applying lemma 5 recursively, we have shown

Proposition 3

$$((\Psi_n, \mathcal{O}_n) \times_{out} \dots \times_{out} (\Psi_1, \mathcal{O}_1), S(\lambda)(\Psi_n, \mathcal{O}_n) \times_{out} \dots \times_{out} (\Psi_1, \mathcal{O}_1)) = (-\mathrm{e}^{\mathrm{i}2\pi\lambda})^{\frac{n(n-1)}{2}} \ . \tag{77}$$

[22] The notion of a right inverse is analogous to that of a left inverse introduced in section 1.

[23] Here $\mathcal{O}_i$ denotes the "support" of Ψ_i, i.e. $\mathbf{B}_i \in \mathcal{F}(\mathcal{O}_i)$, and $\mathcal{O}_i < \mathcal{O}_j$ if $\mathcal{O}_i$ is left to $\mathcal{O}_j$.

Remark: Slightly generalizing the above consideration, let us take different values of λ in the single particle states $\Psi_i = \{\varrho_i, \psi_i\}$, i.e. $\varrho_i = \alpha_{U(1,\lambda_i)}$, such that $\sum \lambda_i = \Lambda$ for fixed Λ. In this case, incoming and outgoing scattering states are linked by

$$S(\Lambda) = (-1)^{\frac{n(n-1)}{2}} \mathrm{e}^{\mathrm{i}\pi(n-1)\Lambda} \; ; \tag{78}$$

put differently: The phase factor occuring in scalar products of scattering states does only depend on the "total charge" Λ.

This result may be interpreted as follows: The morphisms α_U describe free anyons with scattering determined by the statistical parameter. Following a proposal of Swieca [33] to split the S-matrix into a kinematical and a dynamical part, the dynamical one is trivial in our case since the phase factor does not contribute to the scattering cross section.

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